Rigid branchwise-real tree orders

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Supervised by Joel Hamkins

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Summary

- A branchwise-real tree order is a tree where every branch looks like the reals.
- There is a rigid branchwise-real tree order.

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- A branchwise-real tree order is a tree where every branch looks like the reals.
- There is a rigid branchwise-real tree order.
- For $2 \leqslant \kappa \leqslant \mathfrak{c}$, there is a rigid branchwise-real tree order in which every branching point has the same degree κ.
- For $2 \leqslant \kappa \leqslant \mathfrak{c}^+$, there is a rigid branchwise-real tree order in which *every* point has the same degree κ.

Please interrupt to ask questions.

Branchwise-real tree orders

Definition

A *branchwise-real tree order* is a partial order X such that:

1. For every $x \in X$ the set $\downarrow(x) = \{y \in X \mid y \leqslant x\}$ is a linear order.
2. X has a minimum element, its *root*.

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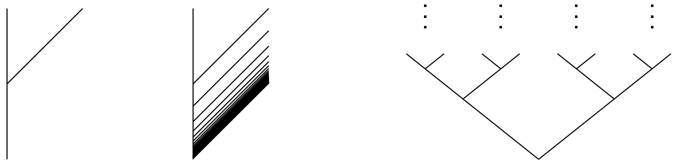
2. X has a minimum element, its *root*.
3. Every two points have a infimum (i.e, X is a meet-semilattice).
4. Every branch (maximal chain) is order-isomorphic to a real interval.



Motivation: $\mathbb{R}$-trees

- An $\mathbb{R}$-tree is a metric space tree in which every point can be branching.
- $\mathbb{R}$-trees are to reals what graph-theoretic trees are to the integers.
- $\mathbb{R}$-trees play an important role in geometric group theory.

[1] Charles Favre and Mattias Jonsson (2004). *The valuative tree*. Lecture notes in mathematics 1853. Berlin; London: Springer. ISBN: 9783540229841.

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Theorem (Favre and Jonsson 2004[1])

The class of underlying orders of $\mathbb{R}$-trees is the same as the class the branchwise-real tree orders which admit a monotonic map into the reals.

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Rigidity

Definition

An *automorphism* of a branchwise-real tree order X is a bijective monotonic function $X \to X$.

Definition

X is *rigid* if it has no non-trivial automorphisms.

Question

Do rigid branchwise-real tree orders exist?

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- Related to group actions on $\mathbb{R}$-trees.
- Analogous to investigation of rigid ω_1-trees; e.g. Jech 1972[2], Fuchs and Hamkins 2009[3].

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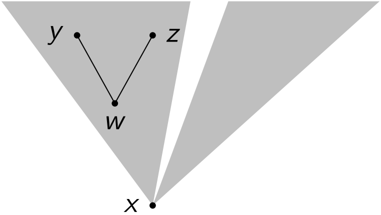
Degrees

Definition

Let X be a branchwise-real tree order and $x \in X$. An *x-connected component* is an equivalence class of $\uparrow(x) \setminus \{x\}$ under the relation:

$$y \sim_x z \quad \Leftrightarrow \quad \text{there is } w > x \text{ such that } w \leqslant y, z$$



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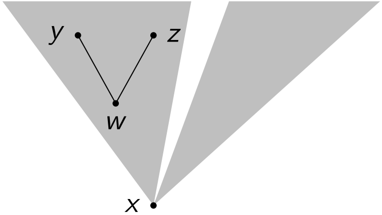

The *degree* of x is the number of x-connected components. It is *branching* if the degree is at least 2.

Branching nodes

- The simple tree which is just one branch is not rigid: $[0, 1]$ has many order-automorphisms.
- If there are $x < y$ in X such that $[x, y] = \{z \in X \mid x \leqslant z \leqslant y\}$ contains no branching nodes, then X is not rigid.
- So branching nodes have to be 'dense' in X.

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- Start with the root.
- At a successor step, above each point added at the previous stage add any number of new ‘spines’ emanating from the root (i.e. copies of $(0, 1)$ or $(0, 1]$).
- At limit steps, new branches through the tree emerge.
- We can decide whether or not to extend these.
- The *rank* of a point is the stage at which it gets added.

A rigid tree

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- We make sure that every point gets a different degree.
- Start with the root.
- At successor stages, for each point x added at the previous stage, pick a new cardinal κ not already used, and add κ-many new spines above x.
- In the resulting tree, every node has a different degree, so there are no non-trivial automorphisms.

Not very elegant

- The resulting tree is huge; its size is the limit of the sequence:

$$\omega_{\mathfrak{c}}, \omega_{\omega_{\mathfrak{c}}}, \omega_{\omega_{\omega_{\mathfrak{c}}}}, \ldots$$

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- What if we wanted to control the size of the tree, or the degree of the branching points?
- Does there exist a rigid tree in which every branching node has the same degree?

A rigid, weakly uniform tree

Definition

A branchwise-real tree order is *weakly κ-uniform* if every branching point has degree κ.

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Theorem (A-D)

Let $2 \leqslant \kappa \leqslant \mathfrak{c}$. There is a rigid, weakly κ-uniform branchwise-real tree order.

Encoding subsets of ω

Proof.

- First construct a collection of dense, mutually non-isomorphic subsets $S_A \subseteq (0, \infty) \mid A \subseteq \omega$, such that S_A encodes A.

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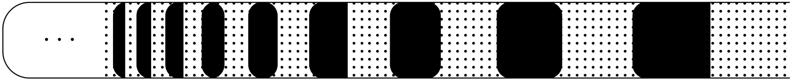
- First construct a collection of dense, mutually non-isomorphic subsets $S_A \subseteq (0, \infty) \mid A \subseteq \omega$, such that S_A encodes A.
- To make S_A, first take $\mathbb{Q} \cap (0, \infty)$.
- Then add a descending sequence of real intervals with limit 0 with irrational extrema, such that the nth interval is open when $n \notin A$ and half-open when $n \in A$.



$$A = \{1, 0, 0, 1, 0, 0, 1, 1, 1, \ldots\}$$

- Think of S_A as a colouring $(0, \infty) \to \{0, 1\}$.

Building the tree

Proof (continued).

- The construction has length ω.
- Start with the root, coloured 1.
- Each time we add a new spine, lay a copy of a different S_A along it.
- Then at the next step, we only add spines above points which get coloured 1. We always add $(\kappa - 1)$-many new spines.

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- Start with the root, coloured 1.
- Each time we add a new spine, lay a copy of a different S_A along it.
- Then at the next step, we only add spines above points which get coloured 1. We always add $(\kappa - 1)$-many new spines.
- The resulting tree X is weakly κ-uniform.
- Let's see that it's rigid.

Why is it rigid? I

Proof (continued).

- Suppose for a ↯ that $f : X \to X$ is a non-trivial automorphism.
- Since each S_A is dense, there is $x \in X$ branching such that $f(x) \neq x$.

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Proof (continued).

- Suppose for a ↯ that $f\colon X \to X$ is a non-trivial automorphism.
- Since each S_A is dense, there is $x \in X$ branching such that $f(x) \neq x$.
- Choose any spine N above x added during the construction.
- Then $f(N)$ is a branch through $\uparrow(f(x))$ and $f{\restriction}N\colon N \to f(N)$ is an isomorphism respecting the colouring.

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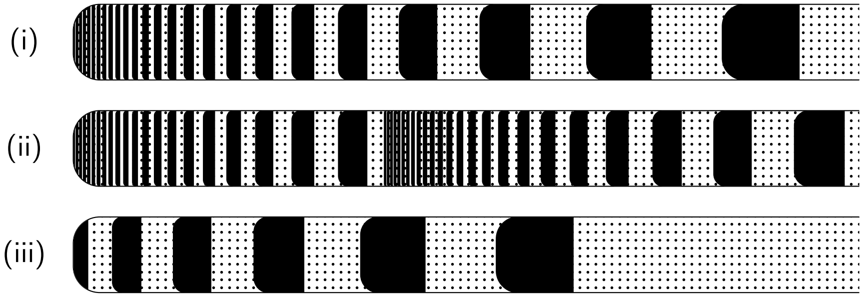
- Then $f(N)$ is a branch through $\uparrow(f(x))$ and $f{\restriction}N : N \to f(N)$ is an isomorphism respecting the colouring.
- The colouring along N looks like a descending sequence of intervals converging at the base, hence the colouring along $f(N)$ must look the same.
- Therefore, $f(N)$ must be a spine added in the construction above $f(x)$.

Why is it rigid? II



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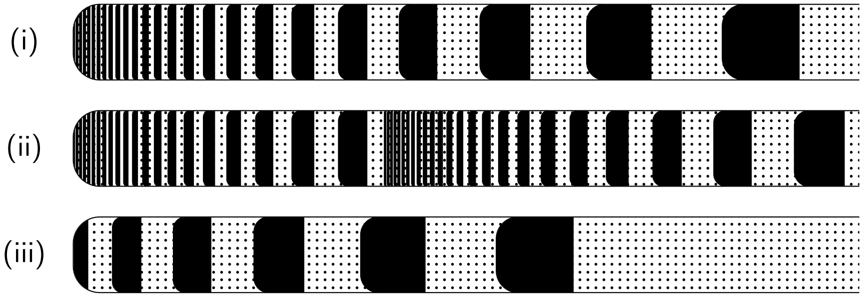

Proof (continued).

- But then N get coloured by S_A, while $f(N)$ gets coloured by S_B, for some $A \neq B$.
- This means there is $y \in N$ such that exactly one of y and $f(y)$ is branching. ↯ □

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Definition

A branchwise-real tree order is *strongly κ-uniform* if every point is branching and has degree κ.

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Let $2 \leqslant \kappa \leqslant \mathfrak{c}^+$. There is a rigid, strongly κ-uniform branchwise-real tree order.

- When we grow the tree, at each successor step we have to add $(\kappa - 1)$-many spines above each point.
- So, if we only make ω-many steps, the resulting tree is homogeneous.
- We need to go higher, and to decide which branches to extend at limit stages.

A generic-esque family of colourings

Lemma (A-D)

Let $\lambda \leqslant \mathfrak{c}$. There is a family $\mathcal{A}$ with size $\mathfrak{c}^+$ of λ-colourings $(0, \infty) \to \lambda$ of the positive real numbers such that the following holds.

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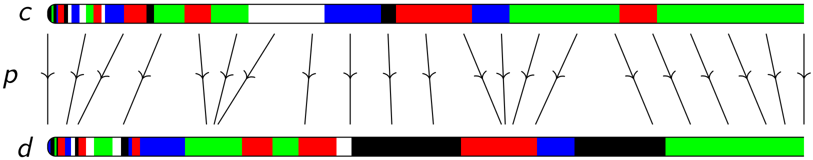

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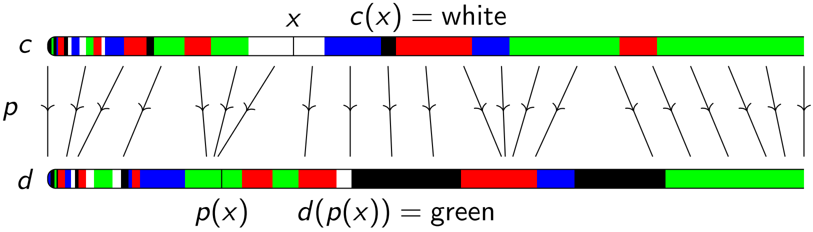

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Proof of Lemma.

- The family is constructed using an iterated diagonal argument.
- By Zorn's Lemma, there is a maximal family $\mathcal{A}$, satisfying both the property in the statement, as well as that for every $c \in \mathcal{A}$:

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- Suppose for a contradiction that $|\mathcal{A}| < \mathfrak{c}^+$.
- We extend $\mathcal{A}$ by diagonalising against the previous colourings, and against every automorphism of $(0, \infty)$.

Proof of the lemma II

Proof of Lemma (continued).

- Enumerate:

$$\mathrm{Aut}(0,\infty) \times \mathcal{A} \times \lambda \times \lambda = \{(p_\theta, c_\theta, \alpha_\theta, \beta_\theta) \mid \theta < \mathfrak{c}\}$$

- Build a new colouring d by diagonalising against these tuples in order.

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- Build a new colouring d by diagonalising against these tuples in order.
- To deal with $(p_\theta, c_\theta, \alpha_\theta, \beta_\theta)$, pick a coinitial sequence (x_n) in $(0,\infty)$ such that each $c_\theta(x_n) = \alpha_\theta$ while $d(p_\theta(x_n))$ is not yet defined, then set each $d(p_\theta(x_n)) := \beta_\theta$.

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- After all this, fill in any remaining undefined parts of d arbitrarily.
- Then $\mathcal{A} \cup \{d\}$ is a larger family. $\lightning$ □

Constructing the tree I

Theorem

Let $2 \leqslant \kappa \leqslant \mathfrak{c}^+$. There is a rigid, strongly κ-uniform branchwise-real tree order.

Proof of Theorem.

- Let $\mathcal{A}$ be a $\mathfrak{c}^+$-sized family of 2-colourings of $(0, \infty)$ as per the lemma.
- We make $\omega \times \omega$-many steps.

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- We make $\omega \times \omega$-many steps.
- At successor steps, we add $(\kappa - 1)$-many spines above points added in the previous stage.
- Along each, lay a different, new colouring from $\mathcal{A}$.
- We use these colourings to decide which branches to extend at limit steps.

Constructing the tree II

Proof of Theorem (continued).

- Take α a limit, and let X'_α be the union of the previous stages, and let $c'_\alpha\colon X'_\alpha \to \{0, 1\}$ be the colouring.

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- A new branch B appearing at the limit contains a piece of rank 1, a piece of rank 2, etc.
- For $\beta < \alpha$ let x_β be maximum element of the piece of B of rank β.

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- For $\beta < \alpha$ let x_β be maximum element of the piece of B of rank β.
- Consider the sequence $(c'_\alpha(x_0), c'_\alpha(x_1), c'_\alpha(x_2) \ldots)$ of colours.
- Add a new point above B iff the sequence has a tail of 1's.

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- Consider the sequence $(c'_\alpha(x_0), c'_\alpha(x_1), c'_\alpha(x_2) \ldots)$ of colours.
- Add a new point above B iff the sequence has a tail of 1's.
- Finally, let X be the $\omega \times \omega$-union of all stages.

Showing that it is rigid I

Proof of Theorem (continued).

- Suppose for a $\lightning$ that $f\colon X \to X$ is a non-trivial automorphism.
- Will only deal with the case $\kappa \geqslant 3$; the $\kappa = 2$ case is more technical.
- There is $y_0 \in X$ such that $f(y_0) \neq y_0$.

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- There is $y_0 \in X$ such that $f(y_0) \neq y_0$.
- Both y_0 and $f(y_0)$ have at least three connected components above, all but one of which contain new spines added above y_0, resp. $f(y_0)$.
- f sends y_0-connected components to $f(y_0)$-connected components.
- There is an y_0-connected component C containing a new spine N such that $f(C)$ contains a new spine M.

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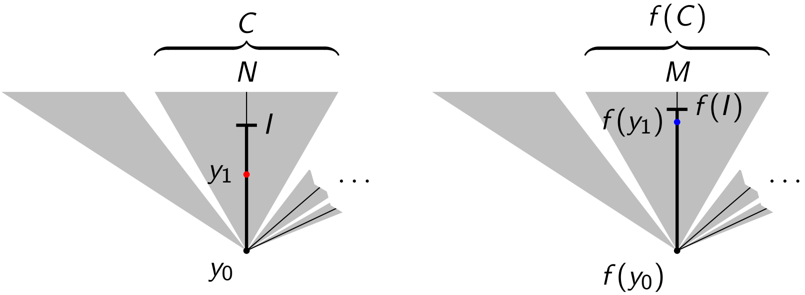
- There is an initial segment $I \subseteq N$ such that $f(I) \subseteq M$.

Showing that it is rigid II



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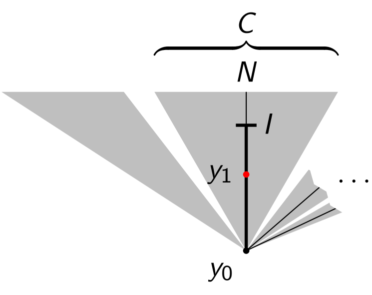

Proof of Theorem (continued).

- The colouring along the spine starting with I is different from that starting with $f(I)$.
- By the key property of $\mathcal{A}$, we can find $y_1 \in I$ coloured 1 such that $f(y_1)$ is coloured 0.

Showing that it is rigid III

Proof of Theorem (continued).

- Iterate this process to get $y_0 < y_1 < y_2 < \cdots$.
- For $n > 0$ we have $c(y_n) = 1$ and $c(f(y_n)) = 0$.

Showing that it is rigid III

Proof of Theorem (continued).

- Iterate this process to get $y_0 < y_1 < y_2 < \cdots$.
- For $n > 0$ we have $c(y_n) = 1$ and $c(f(y_n)) = 0$.
- So at stage $\mathrm{rank}(y_0) + \omega$ the branch determined by $\{y_0, y_1, \ldots\}$ gets extended.
- But at stage $\mathrm{rank}(f(y_0)) + \omega$ the branch determined by $\{f(y_0), f(y_1), \ldots\}$ does not.
- This contradicts that f is an automorphism. $\lightning$ $\square$

Monotonic maps into $\mathbb{R}$

- The tree X admits a monotonic function into $\mathbb{R}$.

Theorem (A-D)

Let $2 \leqslant \kappa \leqslant \mathfrak{c}^+$. There is a rigid, strongly κ-uniform branchwise-real tree order with no monotonic map into $\mathbb{R}$.

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Proof idea.

- This time use ω-colourings of $(0, \infty)$, and continue for ω_1-many steps.
- Extend a branch at a limit step iff no colour appears infinitely often on the sequence of maximal points of each rank.

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- This time use ω-colourings of $(0, \infty)$, and continue for ω_1-many steps.
- Extend a branch at a limit step iff no colour appears infinitely often on the sequence of maximal points of each rank.
- This ensures that in the final tree (i) there is no monotonic map into $\mathbb{R}$, (ii) there are no ω_1-sequences, and (iii) there are no non-trivial automorphisms. □

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- A key technique is to start with a family of colourings of $(0, \infty)$, laying a new one along each spine, and using these to determine how to proceed with the construction.

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Question

Does there exist a rigid, strongly κ-uniform branchwise-real tree order for $\kappa > \mathfrak{c}^+$?

Thanks for listening!

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